TEMPERATURE FIELD IN A TWO-LAYER PLATE HEATED BY A SURFACE SOURCE

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An examination is made of the problem of temperature distribution in a two-layer plate heated by a constant-power surface source.

Surface heating of multilayer plates is widely used in various branches of technology. In the present note we examine the problem of the temperature field in a two-layer plate heated by a constant source of strength q_0. The problem is stated as follows: it is required to find the solution of the equations

$$\frac{1}{a_1}\frac{\partial t_1}{\partial \tau}=\frac{\partial^2 t_1}{\partial z^2} \tag{1}$$

in the region $\tau > 0$, $h \geq z \geq 0$,

$$\frac{1}{a_2}\frac{\partial t_2}{\partial \tau}=\frac{\partial^2 t_2}{\partial z^2} \tag{2}$$

in the region $\tau > 0$, $\infty > z \geq h$, with boundary conditions

$$-\lambda_1\frac{\partial t_1}{\partial z}=q_0 \text{ when } z=0, \tag{3}$$

$$t_1=t_2 \text{ when } z=h, \tag{4}$$

$$\lambda_1\frac{\partial t_1}{\partial z}=\lambda_2\frac{\partial t_2}{\partial z} \text{ when } z=h, \tag{5}$$

$$t_1=t_2=0 \text{ when } \tau=0, \tag{6}$$

$t_2(z,\tau)$ being bounded as $z\to\infty$.

The solution of the system (1)–(6) is found with the aid of a Laplace transformation with respect to τ. We designate

$$\bar{t}(z,s)=\int_0^\infty \exp(-s\tau)\,t(z,\tau)\,d\tau. \tag{7}$$

Applying the transformation (7) to (1), (2) and boundary conditions (3)–(5), solving the differential equations obtained, and using the boundary conditions, we obtain

$$\bar{t}_1=\left(q_0(1+b)\exp\left[\frac{h-z}{\sqrt{a_1}}\right]\sqrt{s}\right)\times\left\{2s\sqrt{s}\left[\frac{\lambda_1}{\sqrt{a_1}}\operatorname{sh}\sqrt{\frac{s}{a_1}}h+\frac{\lambda_2}{\sqrt{a_2}}\operatorname{ch}\sqrt{\frac{s}{a_1}}h\right]\right\}^{-1}+\left[q_0(1-b)\exp\left[\frac{z-h}{\sqrt{a_1}}\right]\sqrt{s}\right]\times\left\{2s\sqrt{s}\left[\frac{\lambda_1}{\sqrt{a_1}}\operatorname{sh}\sqrt{\frac{s}{a_1}}h+\frac{\lambda_2}{\sqrt{a_2}}\operatorname{ch}\sqrt{\frac{s}{a_1}}h\right]\right\}^{-1},$$

$$b=\frac{\lambda_2\sqrt{a_1}}{\lambda_1\sqrt{a_2}}; \tag{8}$$

$$\bar{t}_2=\left(2q_0\exp\left[-\frac{z-h}{\sqrt{a_2}}-\frac{h}{\sqrt{a_1}}\right]\sqrt{s}\right)\times\left\{\lambda_1 s\sqrt{s}\left[\frac{\lambda_1}{\sqrt{a_1}}\operatorname{sh}\sqrt{\frac{s}{a_1}}h+\frac{\lambda_2}{\sqrt{a_2}}\operatorname{ch}\sqrt{\frac{s}{a_1}}h\right]\right\}^{-1}. \tag{9}$$

The inverse transforms of (8) and (9) may be found with the help of the expansion theorem, or by use of a table of inverse transforms [1], after reducing the denominators of (8) and (9) to the form

$$\frac{\lambda_1}{\sqrt{a_1}}\operatorname{sh}\sqrt{\frac{s}{a_1}}h+\frac{\lambda_2}{\sqrt{a_2}}\operatorname{ch}\sqrt{\frac{s}{a_1}}h=\frac{1}{2}\left(\frac{\lambda_1}{\sqrt{a_1}}+\frac{\lambda_2}{\sqrt{a_2}}\right)\times\left[1-g\exp\left(-2h\sqrt{\frac{s}{a_1}}\right)\right]\exp\left(\sqrt{\frac{s}{a_1}}h\right), \tag{10}$$

where $g=(\lambda_2\sqrt{a_1}-\lambda_1\sqrt{a_2})/(\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2})$.

Then for $g > 0$ we obtain

$$\bar{t}_1=\frac{q_0}{s\lambda_1}\sqrt{\frac{a_1}{s}}\sum_{n=1}^{\infty}g^{n-1}\exp\left\{-[2h(n-1)+z]\sqrt{\frac{s}{a_1}}\right\}-\frac{q_0}{s\lambda_1}\sqrt{\frac{a_1}{s}}\sum_{n=1}^{\infty}g^{n-1}\exp\left\{-[2hn-z]\sqrt{\frac{s}{a_1}}\right\}, \tag{11}$$

$$\bar{t}_2=\frac{2q_0}{\lambda_1}\frac{1}{\lambda_1/\sqrt{a_1}+\lambda_2/\sqrt{a_2}}\sum_{n=1}^{\infty}g^{n-1}\times\exp\left[-\left(\frac{z-h}{\sqrt{a_2}}+\frac{h(2n-1)}{\sqrt{a_1}}\right)\right]\sqrt{s}. \tag{12}$$

Using the table of inverse transforms [1], we obtain

$$t_1 = \frac{2q_0}{\lambda_1}\sqrt{\frac{a_1\tau}{\pi}}\sum_{n=1}^{\infty} g^{n-1} \times$$

$$\times\left\{ \exp\left[-\frac{(2h(n-1)+z)^2}{4a_1\tau}\right] - g\exp\left[-\frac{(2hn-z)^2}{4a_1\tau}\right]\right\} - \frac{q_0}{\lambda_1}\sum_{n=1}^{\infty} g^{n-1}\left[(2hn-z)g \times\right.$$

$$\times \operatorname{erfc}\left(\frac{2nh-z}{2\sqrt{a_1\tau}}\right) - \left.(2h(n-1)+z)\operatorname{erfc}\left(\frac{2h(n-1)+z}{2\sqrt{a_1\tau}}\right)\right]. \qquad (13)$$

In the case $g < 0$ the multiplier $(-1)^{n-1}$ must be included under the summation sign in (13), and $|g|$ must be taken. The expression for $t_2(z, \tau)$ is found similarly in the case $g > 0$:

$$t_2 = \frac{2q_0}{\lambda_1\left(\frac{\lambda_1}{\sqrt{a_1}} + \frac{\lambda_2}{\sqrt{a_2}}\right)}\sum_{n=1}^{\infty} g^{n-1}\left\{\exp\frac{1}{4\tau}\left[-\frac{z-h}{\sqrt{a_2}} - \frac{h(2n-1)^2}{\sqrt{a_1}}\right] - \left[\frac{z-h}{\sqrt{a_2}} + \frac{h(2n-1)}{\sqrt{a_1}}\right]\operatorname{erfc}\left[\frac{1}{2}\left(\frac{z-h}{\sqrt{a_2\tau}} + \frac{h(2n-1)}{\sqrt{a_1\tau}}\right)\right]\right\}, \qquad (14)$$

with the remark made for $g < 0$ taken into account.

REFERENCE

1. A. V. Luikov, Theory of Heat Conduction [in Russian], Gostekhizdat, 1952.

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